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Self-Avoiding Walks in Quenched Random Environments

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The self-avoiding walk in a quenched random environment is studied using real-space and field-theoretic renormalization and "Flory" arguments. These methods indicate that the system is described, for $d < d_c = 4$, and, for large disorder for $d > d_c$, by a strong disorder fixed point corresponding to a "glass" state in which the polymer is confined to the lowest energy path. This fixed point is characterized by scaling laws for the size of the walk, $L \sim N^{\zeta}$ with N the number of steps, and the fluctuations in the free energy, $\Delta f \sim L^{\omega}$. The bound $1/\zeta - \omega \leqslant d/2$ is obtained. Exact results on hierarchical lattices yield $\zeta > \zeta_{\text{pure}}$ and suggests that this inequality holds for $d = 2$ and 3, although $\zeta = \zeta_{\text{pure}}$ cannot be excluded, particularly for $d = 2$. For $d > d_c$ there is a transition between strong and weak disorder phases at which $\zeta = \zeta_{\text{pure}}$. The strong-disorder fixed point for SAWs on percolation clusters is discussed. The analogy with directed walks is emphasized.

KEY WORDS: Self-avoiding walks; disordered systems; real-space renormalization group; percolation.

INTRODUCTION

In this paper we discuss the statistical mechanics of a single self-avoiding walk (SAW) in a quenched random environment. This problem has attracted much attention[1–10] in recent years because of its application to polymers in random media, and because it is a simple but nontrivial problem of statistical mechanics with disorder.

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One outstanding question which has generated considerable controversy for more than a decade is whether or not the end-to-end size exponent ζ, of the SAW in the presence of quenched disorder is different from its value for the pure SAW, ζ_0 (often denoted by ν in the literature). Although this question remains unresolved, several authors have recently considered the problem guided by new ideas and methods. In ref. 4, both a real-space renormalization method and a field theory were used to argue that at the percolation threshold $\zeta > \zeta_0$ for $d < 6$. Machta and Kirkpatrick[5] studied the field theory describing the SAW in a Gaussian random environment and showed that disorder leads to runaway trajectories in the renormalization group flow. The RG flows were used to calculate the leading behavior of moments $\langle Z^k \rangle$ of the partition function for $k \geqslant 2$ and $d < 4$; however, the limit $k \to 0$ was not accessible to these methods, so ζ could not be determined. Finally, Obhukov[7] suggested that the statistics of SAWs in quenched random media is described by a strong-disorder fixed point. Evidence for this comes from the fact that in the field theory for $d < 4$ the parameter characterizing the strength of the disorder grows indefinitely under renormalization. This postulated strong-disorder fixed point is, however, inaccessible to conventional perturbative approaches. Using analogies with directed self-avoiding walks (DSAW) with disorder and Flory arguments, Obhukov proposed that $\zeta = \zeta_0$ for $d = 2$, but that $\zeta = \zeta_\perp > \zeta_0$ for $d = 3$ where $\zeta_\perp$ is the wandering exponent for the DSAW. Using similar arguments, Honeycutt and Thirumalai[6] find that $\zeta = 2/3$.

In this paper we investigate the SAW in a random environment using various approaches. A real-space renormalization method, exact on hierarchical lattices, shows that, at least on these lattices, the statistics of the SAWs is indeed controlled by a strong-disorder (or zero-temperature) fixed point characterized by two nontrivial exponents, ζ and ω. We find that ζ is *different* from the pure value, ζ_0, on the same lattice. The exponent ω characterizes the growth of the free energy fluctuations, $\Delta f \sim L^\omega$, with the end-to end-distance L (by analogy with DSAW, we also define $\chi = \zeta\omega$, such that $\Delta f \sim N^\chi$, where N is the number of steps of the SAW). The real-space approach is pursued in Section 2.

Since we do not have analytic methods which permit us to determine the exponents ζ and ω, it is useful to obtain bounds on these exponents. Chayes *et al.*[11] (CCFS) derived a general bound for disordered systems, $\nu \geqslant 2/d$. The proper interpretation of ν for walks in random environments is puzzling. If one naively identifies the correlation length with the size of the walk, this bound is unphysical for $d \leqslant 2$. In Section 3 we show that for SAWs in disordered environments the exponent ζ characterizing the size of the walk is *not* the same as the correlation length exponent, but that the

two are related according to $1/\nu = 1/\zeta - \omega$, which in turn provides a bound. This relation between ν and the "naive" correlation length exponent should be characteristic of zero-temperature fixed points in general (with a nonzero thermal eigenvalue). A similar bound can be obtained for the DSAW.

In Section 4 we consider the problem of the statistics of SAWs on percolation clusters. Using the exact real-space approach, we obtain results which contrast with the approximate real-space calculation of ref. 4 and suggest that a new strong-disorder fixed point controls the SAW statistics at percolation. We then reexamine the field theory of ref. 4 and point out some possible difficulties with the original interpretation of this field theory.

In Section 5 we reexamine the field theory near $d = 4$ and discuss the instability toward strong disorder for $d < 4$. For $d > 4$, we find that small disorder is irrelevant and that, in addition to a strong-disorder regime, a weak-disorder phase exists where the SAW has the statistics of an ideal chain. The phase diagram is determined in a $4 + \varepsilon$ expansion and the exponents of various fixed points are obtained to order ε^2.

Since for the DSAW there is an exact relation, $\chi = 2\zeta_\perp - 1$, between the free energy fluctuations and the transverse wandering, one can also wonder whether there is a similar relation between ζ and ω for SAWs. This is discussed in Section 6, where we also present several scenarios (e.g., Flory arguments, replica scaling, etc.) leading to predictions for the exponents ζ and ω, and their counterpart for the DSAW problem.

Throughout the paper we emphasize analogies with the simpler and better understood problem of the directed self-avoiding walk (DSAW) in a random medium.[12–14] This model has recently received renewed attention due to its importance for understanding the behavior of magnetic flux lines in high-temperature superconductors.[15] We shall see that there are very close similarities with the DSAW problem, which physically is not surprising since by imposing an external field on the isotropic SAW, the SAW is stretched and one recovers the DSAW at large length scales. In Section 7 we study the crossover from SAW to DSAW using scaling or blob arguments similar to those of de Gennes[16] for the pure SAW. We find that the extension L of a SAW along a stretching force F is $L \sim NF^{(1-\zeta)/(\zeta-\chi)}$. By studying the stretched limit, we also argue that the singularity of the free energy of a SAW is characterized by a new exponent, $\nu' = (\omega - 1)/(\omega - 1/\zeta)$. Note that $\nu' = \zeta_0$ for the pure SAW. A similar analysis applied to the DSAW itself yields a simple derivation of the relation $\chi = 2\zeta_\perp - 1$.

The "quenched" problem studied in this work corresponds physically to a polymer in an environment in which one or both ends of the chain are fixed. The "annealed" problem corresponds to averaging the partition function and is associated with a polymer with both ends free which comes to

equilibrium in an infinite environment.[3] The quenched and annealed problems are qualitatively different. In the annealed problem, the polymer finds rare optimal regions in the environment and is *compressed* into those regions. In the quenched problem, the polymer is *stretched* away from the origin in order to find a favorable region. Thus we expect $\zeta_{\text{quenched}} \geqslant \zeta_{\text{pure}} \geqslant \zeta_{\text{annealed}}$. While the annealed problem is reasonably well understood,[2,17,18] the quenched problem is, not surprisingly, much more difficult. Although the physical problem of a polymer in a porous material such as a gel corresponds ultimately to an annealed average, for many purposes the quenched average may be more appropriate. The reason is that the equilibration times[7,19,20] required to find the rare optimal regions dominating the annealed average may be much longer than experimental time scales. On intermediate time scales the polymer explores typical environments and is thus better described by quenched averages.

1. THE MODEL

We define the finite-size free energy for SAWs in quenched random environments as follows. Consider a lattice in a box of volume L^d with an origin and endpoint assigned on opposite faces of the box. Each SAW, Γ, is confined to the box and goes from the origin to the endpoint. For each bond j of the lattice we assign a random energy E_j chosen from a distribution $p(E)$. The statistical weight associated with a SAW occupying the bond j is then $\exp[(-E_j+\mu)/T]$, where T is the temperature and μ the chemical potential for a single step. Note that for fixed $p(E)$, decreasing T amounts to increased disorder while μ shifts the mean energy. Each environment e is defined by a collection of bond energies. The partition function for a given e is the sum over all SAWs of the product of the statistical weights along the SAW

$$Z_L = \sum_{\Gamma} e^{[-E(\Gamma)+\mu N(\Gamma)]/T} \tag{1.1}$$

Here $E(\Gamma)$ is the sum of the energies along Γ and $N(\Gamma)$ is the number of steps in Γ. From the free energy $f_L = -T \log Z_L$ we can obtain the thermal average number of steps $\bar{N} \equiv \bar{N}(\mu, T, e)$ for SAWs in a given environment,

$$\bar{N} = -\partial f_L / \partial \mu \tag{1.2}$$

[3] There is some confusion of terminology here since the environment may be stationary. Since the polymer is free to explore all of space, the partition function is self-averaging. Thus the statistics of a polymer with free ends in a stationary random environment is given by the average of the partition function rather than the average free energy.

For a pure system there is a phase transition as a function of the fugacity $K = e^{\mu/T}$. When $K < K_c$, the chain is stretched and $\bar{N} \sim L$. For $K > K_c$, $\bar{N} \sim L^d$ and the chain is compact. At $K = K_c$

$$\bar{N} \sim L^{1/\zeta_0} \tag{1.3}$$

where ζ_0 is the size exponent for the pure system (e.g., $\zeta_0 = 3/4$ for $d = 2$).

For a random environment, we study the configuration-averaged number of steps $\langle \bar{N} \rangle$. As we vary μ, we observe that $\langle \bar{N} \rangle$ has a stretched phase, a compact phase, and a critical phase at a chemical potential $\mu_c(T, p(E))$ which depends on the temperature and the statistics of the disorder. The size exponent is defined in the critical phase by

$$\langle \bar{N} \rangle = -\partial \langle f_L \rangle / \partial \mu |_{\mu = \mu_c} \sim L^{1/\zeta} \tag{1.4}$$

We emphasize that ζ is associated with the *quenched* problem in which the free energy, rather than the partition function, is averaged. It is reasonable to assume that ζ also characterizes the size $\bar{L}$ of a SAW starting at the origin in a typical environment with a fixed number of steps N on an infinite lattice, $\bar{L} \sim N^{\zeta}$. However, the methods of this paper apply to the exponent ζ as defined in Eq. (1.4).

2. REAL-SPACE RENORMALIZATION GROUP

In the real-space renormalization group approach the Euclidean lattice of the original problem is replaced by a hierarchical lattice. This approach has proven fruitful[21,22] in the study of SAWs in pure environments and has also been successfully used in the study of disordered systems such as directed walks[23,24] and random spin systems.[25,26] In these situations it gives at least qualitatively good results. Here we apply this method to SAWs in random media. Consider SAWs on the hierarchical lattice constructed from the unit cell shown in Fig. 1A. At each step in the construction individual bonds are replaced by unit cells. If there are n levels in the hierarchy, the distance between the endpoints is taken to be $L = b^n$ with $b = 2$. The statistical weight Z' for traversing an n-step structure is a random variable which satisfies the recursion relation

$$Z' = Z_1 Z_2 + Z_3 Z_4 + Z_1 Z_5 Z_4 + Z_3 Z_5 Z_2 \tag{2.1}$$

where $Z_1, \ldots, Z_5$ are statistical weights for the five $(n-1)$-step structures which comprise the bonds of the unit cell and each term in (2.1) arises from a distinct self-avoiding walk across the unit cell.

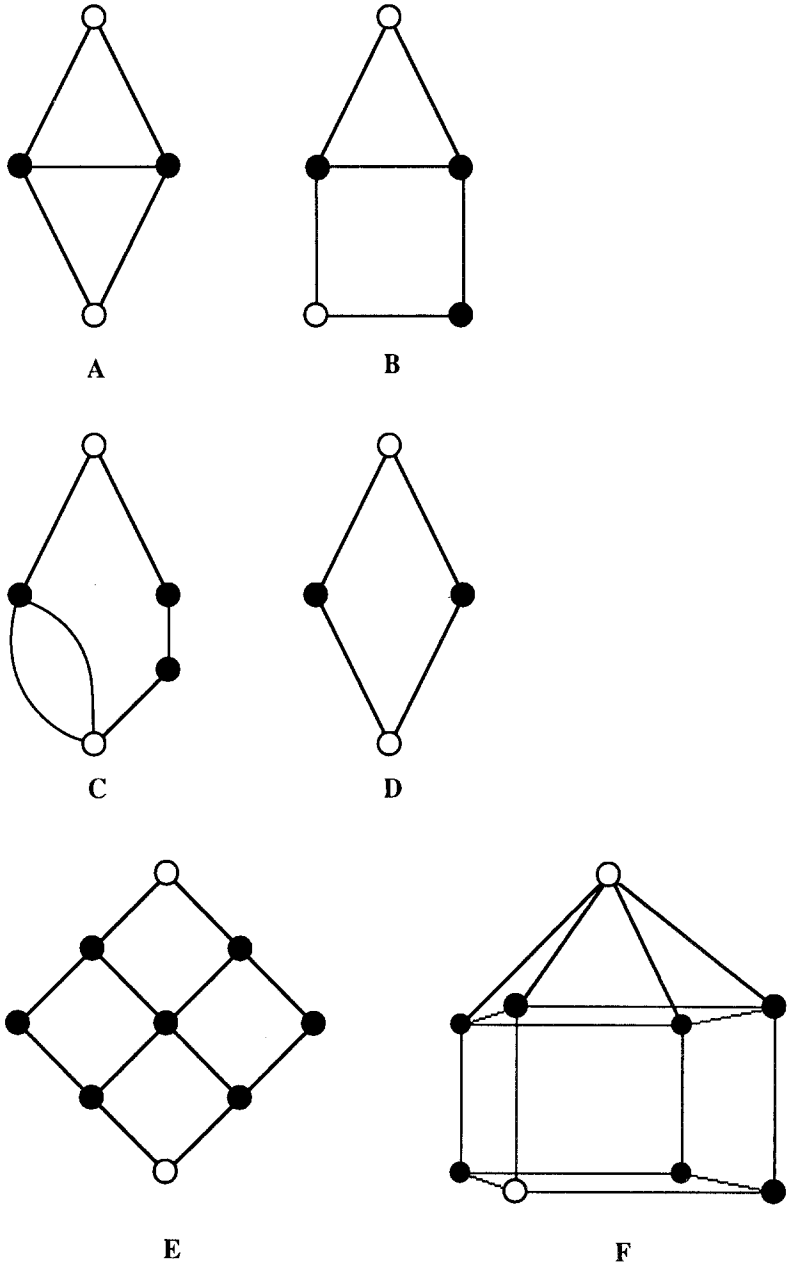

Fig. 1. The unit cells for constructing hierarchical lattices. Self-avoiding walks start at the bottom open circle and end at the top open circle. The length of the bonds between nodes has no significance.

Combining Eqs. (1.2) and (2.1), we find that $\bar{N}$ also enjoys recursion relations,

$$Z'\bar{N}' = (\bar{N}_1 + \bar{N}_2) Z_1 Z_2 + (\bar{N}_3 + \bar{N}_4) Z_3 Z_4 + (\bar{N}_1 + \bar{N}_5 + \bar{N}_4) Z_1 Z_5 Z_4 + (\bar{N}_3 + \bar{N}_5 + \bar{N}_2) Z_3 Z_5 Z_2 \quad (2.2)$$

where the pair $(\bar{N}_j, Z_j)$ is chosen from the joint distribution previously generated by Eqs. (2.1) and (2.2). It is straightforward to construct similar multinomial recursion relations for the other hierarchical lattice shown in Fig. 1, although for lattice F there are 65 distinct SAWs and the corresponding expression is quite lengthy.

For a pure system[21] the statistical weights reduce to a single fugacity variable which satisfies a polynomial recursion relation displaying a phase transition at K_c. At $K = K_c$ we obtain the size exponent ζ_0 according to

$$b^{1/\zeta_0} \equiv \bar{N}'/\bar{N} = \partial K'/\partial K|_{K=K_c} \quad (2.3)$$

For a disordered system we are interested in both the exponent ζ and the exponent ω describing the fluctuations in the free energy,

$$\langle f_L^2 \rangle - \langle f_L \rangle^2 \sim L^{2\omega} \quad (2.4)$$

One can also define $\chi \equiv \omega\zeta$ which characterizes the free energy fluctuations as a function of the number of steps in the SAW. To obtain these quantities, it is convenient to use a Monte Carlo procedure to renormalize the joint distribution $P(f, \bar{N})$. We choose an initial value of μ, T and a large sample of random energies (usually 100,000) chosen from a given distribution. Initially $\bar{N} = 1$ for each bond. Using the recursion relations, we obtain a new sample consisting of pairs of free energies and path lengths. This procedure is very similar to that employed in refs. 23 and 24 to study directed walks. Those authors considered hierarchical lattices, such as shown in Fig. 1D, in which each path has the same number of steps. This leads to a trivial size exponent $\zeta = 1$ appropriate for directed walks where the number of steps in the walk is proportional to the system size in the time direction.

Our first observation is that, for the "two-dimensional" lattices A, B, and E the recursion relations flow toward increasing disorder independent of the choice of μ, T or the initial distribution. This suggests within the real-space approach the idea that SAWs in quenched random environments in sufficiently low dimensionality are controlled by a strong-disorder or "zero-temperature" fixed points. On the other hand, for the "three-dimensional" lattice of Fig. 1F the variance of the free energy distribution decreases if the temperature is high, but increases if the temperature is

sufficiently low. Using the methods of ref. 27, it is straightforward to show analytically that weak disorder is irrelevant, so that in the weak-disorder phase, the distribution flows to the pure fixed point and $\zeta = \zeta_0$. For the hierarchical lattices used here (except for lattice D) the Harris criterion is not satisfied[27] because all bonds are not equivalent. In Section 5 we examine the lower critical dimension for Euclidean lattices d_c above which a phase transition exists between the weak- and strong-disorder phases and conclude that $d_c = 4$.

In the domain of attraction of the strong-disorder phase, the variance of f flows to arbitrarily large values. When the variance of f is very large the largest term in the recursion relations almost always dominates and one can safely replace Eq. (2.1) by the "zero-temperature" recursion relation,

$$f' = \min\{f_1 + f_2, f_3 + f_4, f_1 + f_5 + f_4, f_3 + f_5 + f_2\} \tag{2.5}$$

with similar expressions holding for the other lattices. For a given configuration, $\bar{N}'$ is given by the sum of the $\bar{N}'$ along the path corresponding to the minimum free energy walk. Since the zero-temperature recursion relations are invariant under rescaling of the free energies ($f \to \lambda f$), we can obtain fixed points by rescaling the variance so that it remains fixed under renormalization. By varying the initial value of μ, the distribution flows to one of three strong-disorder fixed points for which the distribution of f is invariant under the recursion relations followed by a rescaling. The three zero-temperature fixed points, two stable and one unstable, correspond to the stretched and compressed phases and the critical point, respectively. These phases correspond to the stretched, compressed, and critical phases of a SAW on a pure lattice.

The fundamental difference between the pure phases and the strong-disorder phases is that in the former case all SAWs of the same length are equally weighted, while in the latter case the ensemble is dominated by the minimum energy path. Indeed, because the problem is dominated by a strong-disorder fixed point one can conjecture that the exponents can be obtained by solving the optimization problem in which the minimum energy SAW of N steps starting at the origin is determined on a random lattice. It has recently been shown that this optimization problem is *NP*-complete.[45]

Under the action of the full recursion relations, there is a critical manifold which connects the pure critical point and the strong-disorder critical point. For a given initial temperature and distribution $p(E)$, the critical manifold is obtained by adjusting μ to a critical value $\mu_c(T, p(E))$. Our object is to determine the exponents associated with the strong-

disorder critical fixed point. If λ_1 is the required rescaling of the free energies at the fixed point, then

$$b^{\omega} = \lambda_1 \tag{2.6}$$

while the size exponent is obtained from the ratio

$$b^{1/\zeta} = \langle \bar{N}' \rangle / \langle \bar{N} \rangle \tag{2.7}$$

evaluated at the fixed point. The values of ζ_0, ζ, ω, χ, and the ratio ζ/ζ_0 are given in Table I for one dimension and the hierarchical lattices shown in Fig. 1. If we suppose that lattices A, B, and E are reasonable approximations to two dimensions, our results suggest that ζ is slightly greater than ζ_0 and that $\chi \approx 0.3$. For the "three-dimensional" lattice F in the strong-disorder phase we find that ζ is substantially larger (9%) than ζ_0. These hierarchical lattice have a fractal dimension $d_f = \log N_b / \log b$, where N_b is the number of bonds in a unit cell which is also given in Table I. Lattices B and F were studied for the pure case in ref. 21.

We note that it is possible to construct hierarchical lattices, such as the one shown in Fig. 1C, for which the random exponent is *less* than the pure exponent, i.e., $\zeta = 0.86 < \zeta_0 = 0.89$. The salient feature of this lattice is that the long path shares no bonds with the two short paths, whereas the short paths share a bond with each other. In comparison to the weighting of paths in a pure system, the minimization defined by the zero-temperature recursion relations favors those paths sharing fewer bonds with other paths. On Euclidean lattices, and faithful representatives such as hierarchical lattices A, B, E, and F, it is the short paths which share the fewest bonds and for these systems we expect $\zeta \geqslant \zeta_0$.

Table I. The Quantities ω, $\chi \equiv \omega\zeta$, ζ, ζ_0, ζ/ζ_0, and d_f for the Hierarchical Lattices Shown in Fig. 1 and for One Dimension[a]

	A	B	C	D	E	F	$d=1$
ω	0.34	$0.44 \pm .01$	—	0.30	0.48	0.46	1/2
χ	0.29	0.32	—	0.30	0.34	0.29	1/2
ζ	0.862	$0.725 \pm .004$	0.86	1	$0.713 \pm .002$	0.64	1
ζ_0	0.847	0.7152	0.886	1	0.7043	0.59	1
ζ/ζ_0	1.018	1.014	0.97	1	1.013	1.09	1
d_f	2.32	2.58	2.58	2	2.39	4	1

[a] The length rescaling factor b is taken as 2 except for lattice E, for which it is $2\sqrt{2}$. Note that χ and ζ/ζ_0 are independent of the choice of b.

3. EXACT RELATIONS AND BOUNDS AT THE STRONG-DISORDER FIXED POINTS

3.1. The exponents ζ and ω

The observation that a properly scaled finite-size free energy goes to a fixed distribution under renormalization allows us to establish a relation between ζ and ω which we believe is exact and applicable to Euclidean lattices. Specifically we will argue that the finite-size scaling correlation length exponent ν defined by Chayes *et al.*[11] (CCFS) is related to ζ and ω via

$$\frac{1}{\nu} = \frac{1}{\zeta} - \omega \tag{3.1}$$

It then follows from the CCFS bound, $\nu \geqslant 2/d$, that

$$\frac{1}{\zeta} - \omega \leqslant \frac{d}{2} \tag{3.2}$$

It should be noted that the CCFS bound is easily generalized to hierarchical lattices if d is replaced by the fractal dimension $d_f = \log N_b / \log b$.

It is instructive to consider the example of SAWs in a one-dimensional random environment. For a chain of length L, the free energy f_L is a random variable which is the sum of L random energies each with variance σ and mean $\langle E \rangle$. According to the central limit theorem, as $L \to \infty$ the distribution for the variable

$$T_L = \frac{1}{\sigma L^{1/2}} [f_L + (\mu - \langle E \rangle) L] \tag{3.3}$$

approaches a Gaussian with mean zero and variance one. T_L can be written in terms of the previously defined parameters ζ, ω, and μ_c,

$$T_L = \frac{1}{\sigma L^{\omega}} [f_L + (\mu - \mu_c) L^{1/\zeta}] \tag{3.4}$$

This identification follows from considering the variance of f_L, which scales as $L^{2\omega}$, and $\langle \bar{N} \rangle = \partial \langle f_L \rangle / \partial \mu$, which scales as $L^{1/\zeta}$. Thus, in one dimension, $\omega = 1/2$, $\zeta = 1$, and $\mu_c = \langle E \rangle$.

In order to identify the finite-size correlation length exponent ν defined by CCFS we must find an appropriate "finite-size scaling event" Y_L. We take the event Y_L to occur when the inequality $-f_L/\sigma L^{\omega} > a$ holds, where

a is some positive constant. The finite-size scaling event must satisfy two properties set forth by CCFS. Property (A) is that there is a positive constant c such that at the critical point $\mu=\mu_c$, $\text{Prob}\{Y_L\}\geqslant 2c$ as $L\to\infty$. This is clearly satisfied since, at the critical point, $-f_L/\sigma L^{\omega}=T_L$ and $\text{Prob}\{-T_L>a\}$ is given by the complementary error function (erfc) with argument a. Property (B) of CCFS is that for $\mu<\mu_c$ and any $\varepsilon>0$, $\text{Prob}\{-f_L/\sigma L^{\omega}>\varepsilon\}$ tends to zero exponentially fast in L. This property is satisfied since the asymptotic behavior of the $\text{erfc}(\varepsilon+(\mu-\mu_c)\,L^{1/\zeta-\omega})$ is exponentially small in L.

The CCFS correlation length $\xi_f(\mu)$ is defined by

$$\xi_f(\mu)=\max\{L\,|\,\text{Prob}[Y_L]>c\} \tag{3.5}$$

The equation which determines the correlation length is then

$$\text{erf}(a+(\mu-\mu_c)\,\xi_f^{1/\zeta-\omega})=2c \tag{3.6}$$

from which we obtain $\xi_f\sim(\mu-\mu_c)^{-\nu}$ with $1/\nu=1/\zeta-\omega$ as promised.

For SAWs in $d>1$ we expect that there is a random variable $X_L=f_L/\sigma^{*}L^{\omega}$ which goes to a fixed distribution as $L\to\infty$ for $\mu=\mu_c$. This distribution will not be a Gaussian, but it must have a finite mean and variance and it must decay sufficiently fast for large values of $|X|$. We have checked this idea qualitatively for hierarchical lattices. The fixed distribution for the hierarchical lattice of Fig. 1B is shown in Fig. 2. If the distribution is scaled so that its variance is one, then the mean is $\langle X\rangle=1.03$. We did not collect enough data to establish the form of the tails of the distribution; however, our results are consistent with an exponential decay of the form

$$P(X)\sim\exp(-a_{\pm}\,|X|^{\delta_{\pm}}) \tag{3.7}$$

where the $+$ and $-$ refer to the sign of X. Plotting $\log|\log P(X)|$ vs. $\log|X|$ yields δ_+ near 2 and δ_- near one.[4]

In one dimension, the distribution for X is a Gaussian for all values of μ and deviations from μ_c appear as shifts in the Gaussian. For $d>1$, the situation is more complicated because the flow away from the fixed distribution is not a simple shift. More generally, we must consider flows

[4] We can estimate the values of δ given the assumption that the tail of the distribution is self-reproducing under renormalization. For $X'\gg\langle X\rangle$ the zero-temperature recursion relations are controlled by the shortest path between the endpoints so that X' is a sum of b random variables, X scaled by b^{ω}. It is straightforward to verify that the tail of the distribution for X is invariant under this operation if $\delta_+=1/(1-\omega)$. For X' large and negative the zero-temperature recursion relations are controlled by the longest path between 0 and L. This path has length L^d. Thus, we obtain $\delta_-=d/(d-\omega)$.

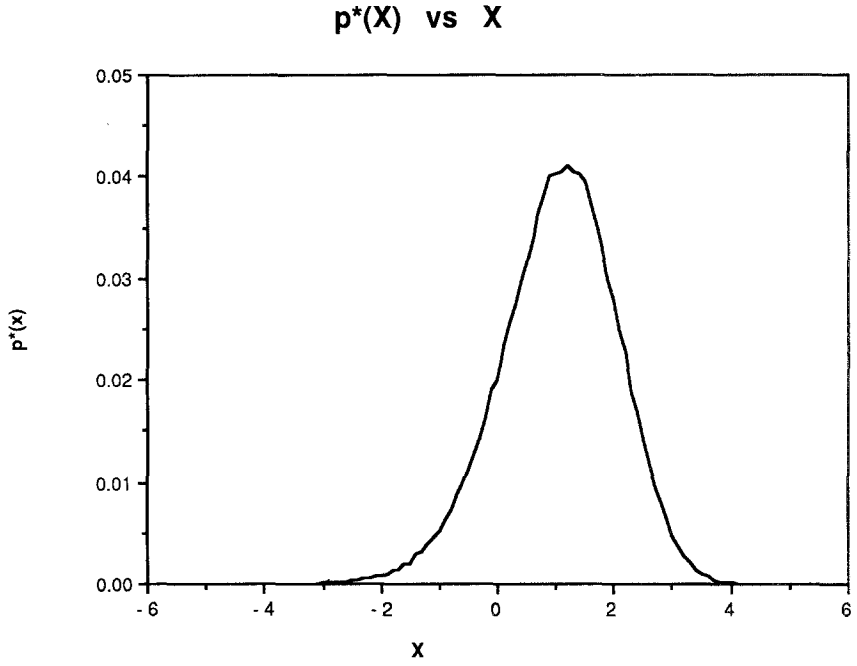

Fig. 2. The fixed distribution $p^*(X)$ for the scaled free energy X for the hierarchical lattice of Fig. 1B. The free energy is scaled so that the variance of p^* is unity.

under renormalization in the space of distributions. The recursion relations such as Eq. (2.5) together with a rescaling by b^{ω} induce a renormalization-group transformation R on the probability density $p(X)$ for the scaled free energy. The fixed distribution p^* satisfies

$$p^* = R[p^*] \tag{3.8}$$

Near the fixed distribution we can linearize the RG transformation so that, if

$$p = p^* + \varepsilon g \tag{3.9}$$

then, for small ε,

$$g' = \mathfrak{R} g \tag{3.10}$$

with $\mathfrak{R}$ a linear operator. We suppose that $\mathfrak{R}$ has one eigenfunction $g^{(1)}$ with eigenvalue larger than unity. We further suppose that if μ deviates from μ_c, then an eigenfunction expansion of p has a nonzero contribution

from $g^{(1)}$. Conventionally, the correlation length exponent ν is identified with the largest eigenvalue of the linearized RG,

$$b^{1/\nu} g^{(1)} = \mathfrak{R} g^{(1)} \tag{3.11}$$

In the region near the fixed point where L is large but $|\mu - \mu_c| L^{1/\nu}$ is small we suppose that only the relevant eigenfunction survives and write the flowing distribution as

$$p(X) \approx p^*(X) + k(\mu - \mu_c) L^{1/\nu} g^{(1)}(X) \tag{3.12}$$

where k is a constant determined by the temperature and the initial distribution. Using the CCFS property A and Eqs. (3.5) and (3.12) and choosing c small, we obtain the finite-size correlation length,

$$\xi_f(\mu)^{-1/\nu} = -\left[k(\mu - \mu_c) \int_a^\infty dX\, g^{(1)}(X) \right] \Big/ c \tag{3.13}$$

Thus, ν is also the CCFS correlation length exponent subject to the bound $\nu \geqslant 2/d$. On the other hand, $\langle \bar{N} \rangle$ is defined from the mean value of the free energy,

$$\langle \bar{N} \rangle = -\partial \langle f_L \rangle / \partial \mu \sim \int L^{1/\nu} g^{(1)}(X)\, L^{\omega} X\, dX \tag{3.14}$$

where the factor $L^{1/\nu}$ arises from the flow of p away from p^* and the factor of L^{ω} from the transformation from X to f. Thus, again we can make the identification $1/\zeta = 1/\nu + \omega$.

3.2. Critical Fugacity

The existence of a strong-disorder fixed point also guarantees that the critical fugacity for the annealed problem is strictly less than for the quenched problem. To see this, we note that the large-L limit of $\langle Z_L \rangle$ can be computed at the quenched critical chemical potential μ_c from the strong-disorder fixed distribution $p^*(X)$ via

$$\langle Z_L \rangle = \int p^*(X)\, dX \exp(-\sigma^* L^{\omega} X / T) \tag{3.15}$$

Assume that p^* has support for $X < 0$, as is the case for the hierarchical lattices studied here; then $\langle Z_L \rangle$ diverges with L at μ_c. On the other hand, it is straightforward to verify from (1.1) and the existence of the pure

critical point that there is an annealed critical point $\mu_{c,\text{annealed}}$ at which $\langle Z_L \rangle$ is finite as $L \to \infty$,

$$\mu_{c,\text{annealed}} = \mu_{c,0} - T \log \langle \exp(-E/T) \rangle \tag{3.16}$$

where $\mu_{c,0}$ is the pure critical point and E is a single-site (bond) energy. Since $\langle Z_L \rangle$ increases with μ, we see that

$$\mu_c > \mu_{c,\text{annealed}} \tag{3.17}$$

Supposing that the critical fugacities are proportional to the connective constants in the fixed-N representation, we have the corresponding strict inequality for large N,

$$\lim_{N \to \infty} \log \langle Z_N \rangle / \langle \log Z_N \rangle > 1 \tag{3.18}$$

where Z_N is the partition function for SAWs of N steps starting at the origin. Equation (3.18) agrees with recent simulations.[10]

4. SELF-AVOIDING WALK ON PERCOLATION CLUSTERS

A diluted Euclidean lattice is a particular case of a quenched random environment. Let p be the fraction of bonds present in the lattice, and p_c the bond percolation threshold. If $p > p_c$ and if universality holds, we expect the statistics of the SAW to be controlled by the same strong-disorder fixed point as discussed above. The physical reason is the following. For $p > p_c$ and at scales larger than the percolation correlation length, the lattice still resembles a Euclidean lattice (link-node-blob picture), although with some local irregularities. At large length scales there is an effective random potential generated as a purely entropic effect. At $p = p_c$, however, the problem is different and amounts to studying the statistics of SAWs on percolation clusters. In that case too, despite much work,[2–4,8,9,28,29] there remains considerable disagreement on whether or not the exponents are different from those for the pure system. The most recent numerical work[3,8] suggests that in $d = 2$ at $p = p_c$ the size exponent is very close or equal to its pure system value. In a recent study, Meir and Harris[4] have applied both real-space RG and field theory methods to this problem. Their results are that ζ is increased by disorder at $p = p_c$ below $d_c = 6$ and they obtain a $6 - \varepsilon$ expansion of ζ on Euclidean lattices. In this section we consider this problem, using the real-space RG methods of Section 5 applied to the same hierarchical lattice as ref. 4. Thus, we follow the full distribution of bond free energies rather than the approximate bimodal distribution of ref. 4. Our approach is exact on this particular hierarchical lat-

tice, except for statistical errors in following the distribution. Although we also find that ζ is changed, we find that the statistics of SAWs is controlled once again by a strong-disorder fixed point, different from the one for $p > p_c$. The strong-disorder nature of this fixed point is out of reach of the method used in ref. 4, since these authors project onto a bimodal distribution at each RG step, an approximation which is not justified since the width of the distribution of nonzero bonds becomes arbitrarily large.

In this section we also comment on the field theory approach to this problem used in ref. 4. Although the situation is as yet unclear, we point out that there is a possibility that the analysis in ref. 4 may not describe SAWs on the percolation cluster.

4.1. RSRG Approach

We use the hierarchical lattice of Fig. 1A. The distribution of the Z_i associated with each bond is now of the form $P(Z) = (1-p)\,\delta(Z) + pQ(Z)$, and one has to follow both the evolution of the percolation concentration p as well as the "continuous" part $Q(Z)$ under the recursion relation (2.1), which we recall here:

$$Z' = Z_1 Z_2 + Z_3 Z_4 + Z_1 Z_5 Z_4 + Z_3 Z_5 Z_2 \tag{4.1}$$

In this approach we average only over those configurations which support at least one SAW from the origin to the endpoint of the lattice, that is, all averages are with respect to $Q(Z)$. Although the initial form of $Q(Z)$ is $Q(Z) = \delta(Z - Z_0)$, it does not remain of this form and rapidly becomes very complicated. The recursion relation for p decouples:

$$p' = f(p) = 2p^2 + 2p^3 - 5p^4 + 2p^5 \tag{4.2}$$

$p = p_c = 1/2$ is the unstable percolation fixed point of Eq. (4.2) and for $p > 1/2$, $p \to 1$ the pure value. We have used a method analogous to the one of Section 2 to follow $Q(z)$. The flow diagram is displayed schematically in Fig. 3, where we show the three relevant variables, p, var(log Z) (or alternatively T^{-2}), and $\langle \log Z \rangle / [1 + \mathrm{var}(\log Z)]^{1/2}$. In particular, there are now *two* strong-disorder fixed points, one (SD_{p_c}) for $p = p_c$ which controls the SAW on the percolation cluster, and the one studied in Section 2 (SD_1) controlling the SAW on an undiluted random lattice, $p = 1$, with disordered initial $Q(Z)$. It is important to notice that the plane of initial conditions var(log Z) = 0, which was represented in Fig. 1 of ref. 4, is unstable and one

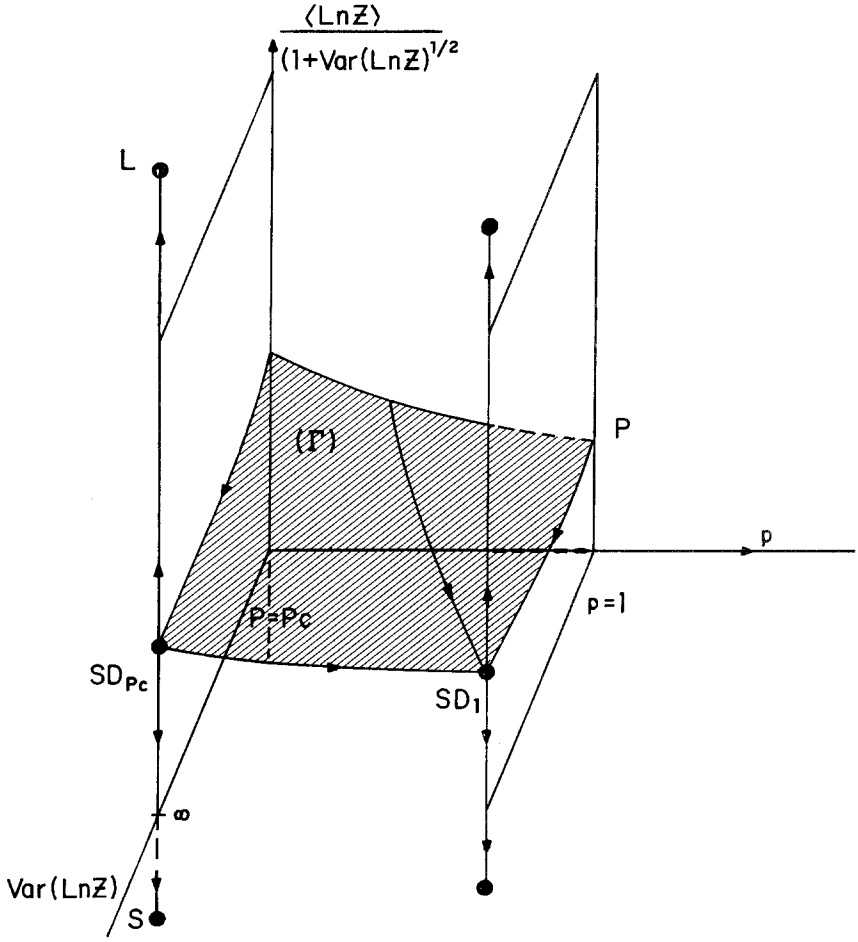

Fig. 3. Schematic flow diagram for SAWs on the diluted hierarchical lattice of Fig. 1A. The fixed points are as follows: (P) pure critical point; (SD_1) strong disorder for $p > p_c$; (SD_{p_c}) strong-disorder fixed point at the percolation threshold; (L) longest path on a percolation cluster; (S) shortest path on a percolation cluster.

immediately flows outside this plane. The critical manifold (Γ) which contains the two strong-disorder fixed points and its intersection with this initial plane is also represented. The fixed points representing the longest (L) and shortest (S) SAWs on percolation clusters are also represented.

The strong-disorder fixed point (SD_{p_c}) at percolation $p = p_c$ is then studied and as in Section 2 one can use zero-temperature recursion

relations in the following form, by enumerating all possible cases and exploiting the symmetries (all the f_i are independently distributed):

$$
\begin{aligned}
&f' = f_1 + f_2 \quad \text{with probability } 1/2 \\
&f' = f_1 + f_2 + f_3 \quad \text{with probability } 1/8 \\
&f' = \min(f_1 + f_2, f_3 + f_4) \quad \text{with probability } 1/16 \\
&f' = \min(f_1 + f_2, f_1 + f_5 + f_4) \quad \text{with probability } 1/4 \\
&f' = \min(f_1 + f_2, f_3 + f_4, f_1 + f_5 + f_4, f_3 + f_5 + f_2) \quad \text{with probability } 1/16
\end{aligned}
\tag{4.3}
$$

The probabilities are obtained using $p_c = 1/2$, and N' is given by the sum of the N along the minimizing path in each case. The rest of the analysis is identical to Section 2. We find $\zeta = 0.850$. The value obtained using the approximate method of ref. 4 at the fixed point B is $\zeta' = (\log 2/\log \lambda_K) \simeq 0.8488$ (note that these authors choose to make a different identification of ζ using the exact value of the percolation correlation length exponent on 2D Euclidean lattices, but this is irrelevant here since we are really only comparing the *eigenvalues*). Although these values are close, they are definitely distinct. They are also distinct from $\zeta_0 \simeq 0.8465$ on the same lattice, and from $\zeta = 0.86$ at the strong-disorder fixed point for $p_c < p < 1$. In addition, we find $\omega(p = p_c) = 0.563$.

Let us make some comments now on the shortest and longest paths. The corresponding exponent for the shortest path is given by studying the recursion relations: $N' = N_1 + N_2$ with probability 7/8, $N' = N_1 + N_2 + N_3$ with probability 1/8, and all $N_i = 1$ as an initial condition. By studying the transformation rules for the first moments of the distribution of N's one easily shows that the variable $u = Nb^{-1/\zeta_{sp}}$, with $\zeta_{sp} = \ln 2/\ln(17/8) \simeq 0.9196$, has a nontrivial limit distribution with variance $\sigma = 7/153$ and average 1. Higher moments could also be computed analytically. Similarly, for the longest path one finds $\zeta_{lp} = \ln 2/\ln(39/16) \approx 0.778$. One has the rigorous bounds $\zeta_{lp} < \zeta_{SAW}(p = p_c) < \zeta_{sp}$. These exponents were previously studied[30] numerically on the same lattice, yielding results fairly close to our exact values, although the error bars in ref. 30 were too optimistic. It is surprising that these exponents can be found analytically while the exponents associated with the *free energy* are controlled by a strong-disorder fixed point and cannot be obtained analytically. For instance, for very large negative f_i (e.g., K_i very small—point E in ref. 4) the recursion relation (4.3) becomes $f' = \min\{f_1 + f_2, f_3 + f_4\}$ with probability 1/8, $f' = f_1 + f_2$ with 3/4, $f' = f_1 + f_2 + f_3$ with 1/8. This leads to ζ_{sp} for the lengths of the path since the recursion relations for f and for N are decoupled. However, due to multiply connected paths, the fixed distribution for the free energy is nontrivial, characterized by a nontrivial ω.

4.2. Field Theory

Meir and Harris[4] wrote down a field theory describing SAWs on percolation clusters which has a perturbatively accessible fixed point around $d=6$. They studied the partition function (i.e., susceptibility)

$$F(p, K)=\sum_j \langle v_{ij} \log Z(i, j, K)\rangle$$

where $K\equiv e^{\mu/T}$ and $v_{ij}=1$ if i and j are connected and zero otherwise. From $F(p, K)$ one can compute the double average $\langle \bar{N}_{ij}\rangle$ by differentiation with respect to $\log K$. This field theory is very similar to the one used in the study of the dilute XY model or the random resister network[31] and using techniques developed for these problems, they obtain one-loop RG recursion relations for the spatial Fourier transform $G_k(q)$ of the replica moments $\langle v_{ij} Z(i, j, K)^k\rangle$. Denoting $G_k(q)=(q^2+r_k)^{-1}$, they find that the variables r_k enjoy[4] the following recursion relations [with $G_k\equiv G(1)$] to order ε:

$$b\frac{dr_k}{db}=(2-\eta)\, r_k+\frac{2\varepsilon}{7} G_k G_0-\frac{\varepsilon}{7}\sum_{s=0}^{\infty}\frac{k!}{(k-s)!\, s!} G_s G_{k-s} \tag{4.4}$$

with $\eta=-\varepsilon/21$.

The variable r_k must then be expanded around $k=0$, $r_k=r_0+u_1 k+u_2 k^2+\ldots$, and one is interested in the recursion relation of the coefficients r_0, u_1, u_2,.... The variable r_0 is the usual percolation variable $\sim(p-p_c)$, and u_1 is related to the variable $\langle \log Z\rangle$ and u_z with Var($\log Z$). From (4.4) one obtains

$$b\frac{du_1}{db}=\left(2-\eta-\frac{\varepsilon}{7}\right) u_1+\frac{\varepsilon}{7}\sum_{s=1}^{\infty}\frac{1}{s}(-)^s G_s G_{-s} \tag{4.5}$$

The authors of ref. 4 then argue that Eq. (4.5) is of the form $bd(u_1-u_{1c})/db=(u_1-u_{1c})/\zeta$, and thus identify ζ as $\zeta=1/2+\varepsilon/42$.

We claim, however, that one must be cautious here in interpreting the field theory. There is a fixed point of (4.4) to order ε, namely

$$r_k^c=\frac{\varepsilon}{7}(2^{k-1}-1) \tag{4.6}$$

obtained from (4.5) by supposing all r_k of order ε. At this fixed point the above identification of the size exponent is certainly correct. The problem is that this fixed point has infinitely many unstable directions. It is unstable not only in the directions r_0 and u_1, as is expected to describe the SAW,

but also along the directions u_2, u_3, etc., as one easily sees by deriving recursion relations for u_2, u_3 analogous to (4.5). It remains to be shown that these directions are actually irrelevant to describe the SAW. The existence of an infinity of *a priori* relevant directions is a well-known feature of the ordinary percolation fixed point (which is a multicritical fixed point), and is also encountered in the problem of the resistor network (respectively the XY model). However, there is an important difference. In these problems there is no order at percolation. The conductivity of the resistor network (respectively the magnetization of the XY model) vanishes at $p = p_c$. One then starts from the percolation fixed point $u_{kc} = 0$, and adds the u_k with $k \geqslant 1$, which are thus *perturbations all proportional* to σ^{-k} (K^k respectively) of the percolation fixed point, with σ the conductance of the bonds of the resistor network (K^{-1} is the temperature of the XY model). What is studied is the *crossover* when these perturbations are turned on. It is then explicitly shown[31] that u_2, u_3,... lead only to correction to the crossover exponent because if $u_n \sim u_1^n$ initially, it remains true under renormalization, and thus, up to subdominant contributions, renormalization simply amounts to a rescaling of u_1, e.g., σ or K^{-1} respectively [see Eq. (2.21) of ref. 31]. Here there is a transition at percolation at a *finite* "temperature" K_c^{-1} which one can reach by varying $\mu = \log K$. In these other models the transition was at *zero* "temperature," e.g., $\mu = -\infty$ or $K^{-1} = 0$ (this "temperature" K^{-1} appears if these models are expressed in terms of spin models, but is unrelated to the variable T for the SAW introduced in Section 1). Thus, the nature of the problem is different, and here one needs to find a fixed point with *only* two unstable directions describing the transition at $K = K_c$ and $p = p_c$.

In the present case $r_k \equiv r_0^c$ is thus no longer a fixed point (although r_0^c coincides with its value for percolation $-\varepsilon/14$) and thus (4.6) is a new fixed point different from the percolation fixed point. Since $G_s G_{-s}$ is even in s, the sum in (4.5) depends only on u_1^2. The recursion relation (4.5), at least to order ε, takes the form $d(u_1 - u_{1c})/dl = (u_1 - u_{1c})/\zeta$. One could then argue that, since it is decoupled from the others, and since we are only interested in the behavior of $\langle \log Z \rangle$, the fact that the u_k, $k > 1$, flows away is without consequence for the exponent ζ. This seems to be true for $d \geqslant 6$, where $\zeta = 1/2$ due to the "geometric" constraint $1/2 = \zeta_{\text{lp}} \leqslant \zeta \leqslant \zeta_{\text{sp}} = 1/2$. However, for $d < 6$, one sees from (4.5) that u_{1c} depends *a priori* on all the other variables in a complicated way, and can be considered as constant to $O(\varepsilon)$ only as long as all the u_k remain of order ε. This can be the case, it seems, *only* if all initial parameters $u_k > 1$ sit exactly at the fixed point; otherwise, they quickly flow away and u_{c1} cannot be considered as constant. Since one can always vary the microscopic parameters of the model, or also introduce a small disorder in the K within

the cluster, there is no reason in general for the initial values of the u_k, $k \geqslant 2$, to be at the fixed point (4.6).

To conclude, the instability of the fixed point (4.6), and in particular the fact that u_2 grows unless exactly at the fixed point, could indicate that the critical behavior of SAWs at $p = p_c$ for $d < 6$ is described by another fixed point, presumably at strong disorder (large u_k, $k \geqslant 2$). Further analysis of the field theory is called for to determine whether the result for ζ to $O(\varepsilon)$ proposed in ref. 4 depends or not on the behavior of the u_k.

5. FIELD-THEORETIC APPROACH NEAR FOUR DIMENSIONS

5.1. Review of the Field Theory

The quantity $\langle Z(0, x, K)^k \rangle$ can be represented by a field theory[5,7] with fields φ_i^α, $i = 1,..., n$ and $\alpha = 1,..., k$, in the limit $n \to 0$:

$$\langle Z(0, \xi, K)^k \rangle = \int D\varphi_i^\alpha \prod_{\alpha=1}^{k} \varphi_i^\alpha(0)\, \varphi_i^\alpha(x) \exp(-S_k)$$

$$S_k = \frac{1}{2} \int dx \sum_{\alpha=1}^{k} [(\nabla\varphi^\alpha) \cdot (\nabla\varphi^\alpha) + t\varphi^\alpha \cdot \varphi^\alpha + u(\varphi^\alpha \cdot \varphi^\alpha)^2]$$

$$- \frac{\Delta}{8} \sum_{\alpha \neq \beta}^{k} (\varphi^\alpha \cdot \varphi^\alpha)(\varphi^\beta \cdot \varphi^\beta)$$

t is the reduced temperature and Δ is proportional to the variance of the random potential. The true self-avoidance parameter is U and the parameter $u = U - \Delta/8$ has been defined for convenience (thus, $u = -\Delta/8$ corresponds to an ideal chain in a random potential).

In terms of renormalized quantities the RG recursion relations were shown to be[5,7]

$$b \frac{dt}{db} = (2 - 4u)\, t$$

$$b \frac{du}{db} = \varepsilon u - 32u^2 \tag{5.1}$$

$$b \frac{d\Delta}{db} = \varepsilon\Delta + 2\Delta^2 - 16\Delta u$$

where $\varepsilon = 4 - d$ and b is the renormalization scale. One sees[5] from (5.1) that (i) the recursion relations for u and t are decoupled from the disorder

and thus that $u>0$ goes to u^*, the value at the pure SAW fixed point (this feature persists at any finite order in perturbation theory due to the limit $n\to 0$) and (ii) for $d<4$, Δ grows indefinitely under RG and there is no stable fixed point.

From (5.1) one can first identify a length scale b_1 obtained by setting $\Delta(b_1)\sim 1$, such that perturbation theory can be trusted only for $b\ll b_1$. One has $b_1=b_0\exp[(2\Delta_0)^{-1}]$ for $d=4$, and for $d<4$, $b_1=b_0(1+\varepsilon/4\Delta_0)^{2\varepsilon}$, where Δ_0 need not be small compared to ε [we have set $u(b)$ at its fixed-point value u^*].

It was argued in ref. 5 that for $k>1$, there is another length scale b^* at which a *first-order* transition takes place where the φ fields acquire a nonzero expectation value. This occurs at $\Delta(b^*)=\varepsilon/[4(k-1)]$, which gives $b^*=b_1 k^{-\varepsilon}$. Clearly this can be trusted only for $k-1\gg\varepsilon/4$. The first-order transition occurs at the "temperature" $t_R=t_c^k\sim\varepsilon(k\Delta_0/\varepsilon)^{4/\varepsilon}>0$, where $t_R=0$ is the second-order transition temperature. From this one can deduce[5] that

$$\langle Z_N^k\rangle\sim\langle Z_N\rangle^k\exp(kt_c^kN) \tag{5.2}$$

5.2. Comments on the Field Theory, $4+\epsilon$ Expansion

We would like now to add some remarks to the analysis of ref. 5. There are several similarities between the recursion relations (5.1) around $d=d_c=4$ and those which describe directed self-avoiding walks or equivalently the KPZ equation[12,13] around $d'=d'_c=2$, where d' is the number of transverse dimensions. To first order in ε the recursion relations are[12,13,15]

$$b\frac{dD}{db}=(-2+z)D,\qquad b\frac{d\Delta}{db}=\varepsilon\Delta+\Delta^2 \tag{5.3}$$

with $\varepsilon=2-d'$, D is the diffusion coefficient of one walk, z the dynamical exponent, and Δ the strength of the disorder. The first equation would be similar to $b\,dt/db=2t$. This similarity in the structure of the RG flow around the pure fixed point is even more striking when several directed walks with mutual repulsion are considered, as in ref. 15, for which one has the additional recursion relation for the mutual repulsion strength u ($u=U$ here)

$$b\frac{du}{db}=\varepsilon u-u^2 \tag{5.4}$$

The similarities are the following, as one can see by comparing formulas (5.1) and (5.3), (5.4): (i) the disorder is decoupled from D (and u)

and (ii) Δ grows indefinitely. The main difference of course is that for the directed polymer, due to exact Galilean invariance, D is unrenormalized (for an appropriate definition of D this feature is exact) and does not couple to u.

We would like to emphasize that, although the recursion relations (5.1) are relative to the Hamiltonian of the replicas, they *do not* depend on the number of replicas. They also yield the recursion relations for the renormalization of the *physical* parameters of the model, quite independently of any considerations involving replicas.[14] They can be derived directly without replicas,[14] a procedure which usually is equivalent to doing the limit $k \to 0$ in the recursion relations.

It is now well established in the case of the directed walks that it would be *incorrect* to conclude from Eq. (5.3) that $\zeta = 1/2$. The reason is that there is no fixed point at small positive Δ for $d' < 2$ and that, as other approaches show, the physics is controlled by a strong-disorder, or zero-temperature fixed point (at $\Delta = \infty$).[12-14,23] At this fixed point $\zeta > 1/2$ ($\zeta = 2/3$[33] in $d' = 1$). Thus, we believe it would be dangerous to conclude similarly that $\zeta = \zeta_0$ from the fact that the recursion relations for the SAW close to the pure fixed point are unchanged by disorder. It seems reasonable instead to conclude from the field theory and the analogies with the DSAW that the presence of runaway trajectories for the SAW below $d = 4$ indicates that the system is controlled by a strong-disorder fixed point, an idea first stated by Obhukov.[7] It is possible that $\zeta = \zeta_0$ at this new fixed point, but in that case one expects to find an underlying reason such as a symmetry.

Let us now take a closer look at the RG flow diagrams. For $d > 4$ ($\varepsilon < 0$) it is clear that there exists a finite $\Delta = \Delta_c$ perturbative fixed point to (5.1) as sketched in Fig. 4. There is a phase where both disorder and self-avoidance are irrelevant and where the polymer behaves like an ideal chain. The frontier of this domain can be determined within perturbation theory for $|\varepsilon|$ small (as, for instance, in the $2 + \varepsilon$ expansions of the nonlinear sigma model[34]). We have represented the complete domain $\Delta \geqslant 0$ and arbitrary u. It is important to remember that $u \equiv U - \Delta/8$, where U is the "true" self-avoidance of the polymer. On the invariant line $u = -\Delta/8$ ($U = 0$), which corresponds to the ideal chain in a random potential,[35] there is a multicritical fixed point M. For $u > -\Delta/8$ ($U > 0$) the frontier of the phase where both u and Δ are irrelevant is a curve $\Delta_c(u)$ identical to the stable manifold of the fixed point C. The fixed point C thus controls the transition from this phase to a strong-disorder regime. Similarly, M controls only the transition for $U = 0$ between strong and weak disorder. Finally, for $u < -\Delta/8$ (attractive interaction) the frontier is a vertical line $u = -|\varepsilon|/32$, controlled by S. Note that in $d > 4$ random walks need a finite amount of

attraction to collapse (for $u < -|\varepsilon|/32$). The corresponding flow diagram for $d < 4$ is represented in Fig. 5. The fixed point S is now at $u = \varepsilon/32 > 0$ and corresponds to pure SAWs. There are no other nontrivial perturbative fixed points, since M and C are now in the "unphysical" regime $\Delta < 0$.

Since C corresponds to a finite and small Δ_c it is now correct to conclude from the decoupling property mentioned above that $\zeta = 1/2$ on the boundary between weak and strong disorder. A similar result holds[14] for directed walks at $\Delta = -\varepsilon$. In that problem, however, it was argued[14] that, due to Galilean invariance, it must be true exactly. Here the argument is only perturbative in ε. Despite this fact, the fixed point C has some features which differ from the Gaussian fixed point. As is the case for directed walks,[14] the functuations of the free energy should grow like some power of $\log N$. We have not yet attempted to compute these fluctuations.

The exponents associated with the fixed points S, M, C have been studied to order ε^2 before[32] (more generally for an m-vector model). However, M and C (for $m = 0$) have been termed unphysical.[32] Since here,

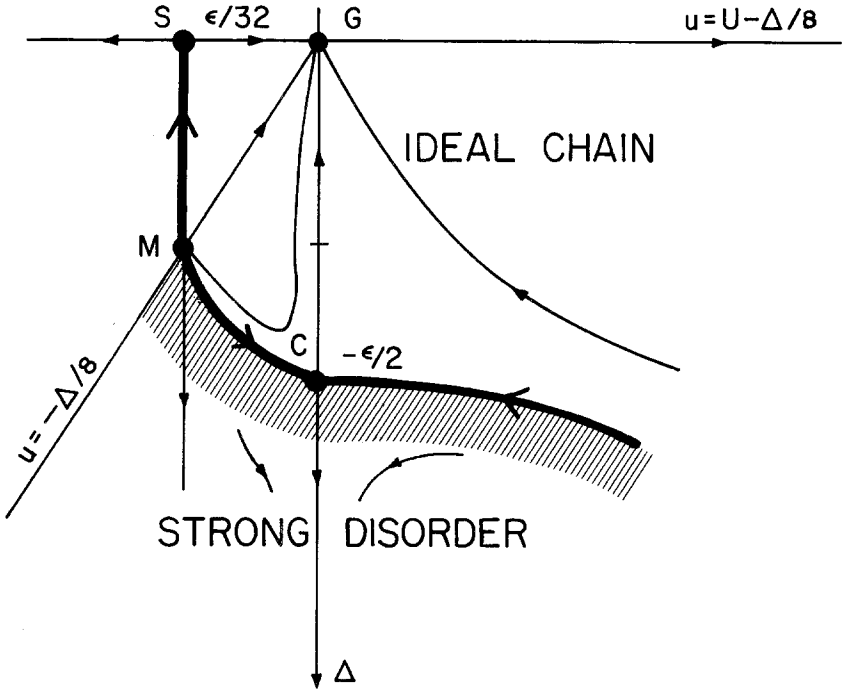

Fig. 4. Field-theoretic renormalization group flow diagram for $d > 4$.

in $d>4$, they have an interesting physical interpretation, we give their associated exponents[32] (only for $\varepsilon<0$ for M and C, and for any ε for S).

$$S: \quad \nu=\frac{1}{2}+\frac{1}{16}\varepsilon+\frac{15}{4}\varepsilon^2, \quad \eta=\frac{1}{32}\varepsilon^2, \quad \alpha=\frac{1}{4}\varepsilon-\frac{7}{128}\varepsilon^2$$

(S = self avoiding walk for $\varepsilon=4-d>0$, self-*attracting* walk *at* the collapse transition for $\varepsilon=4-d<0$);

$$M: \quad \nu=\frac{1}{2}-\frac{1}{16}|\varepsilon|-\frac{27}{512}\varepsilon^2, \quad \eta=\frac{1}{64}\varepsilon^2, \quad \alpha=-\frac{1}{4}|\varepsilon|-\frac{35}{128}\varepsilon^2$$

(M = ideal chain in a random potential *at* the transition weak–strong disorder for $d=4+|\varepsilon|$);

$$C: \quad \nu=\frac{1}{2}, \quad \eta=-\frac{1}{32}\varepsilon^2, \quad \alpha=-\frac{1}{2}|\varepsilon|$$

(C = self-avoiding walk in a random potential *at* the transition weak–strong disorder for $d=4+|\varepsilon|$). Here one can make the identification $\zeta=\nu$.

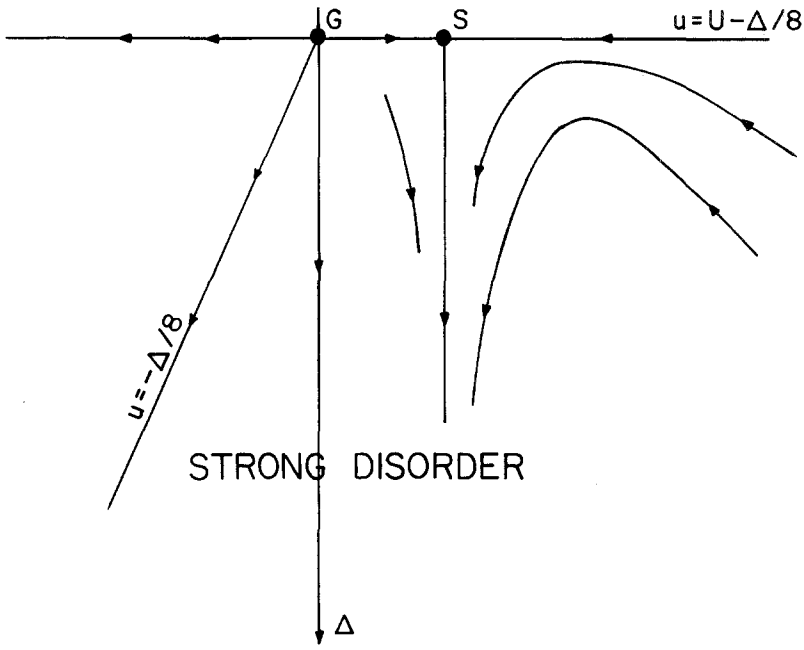

Fig. 5. Field-theoretic renormalization group flow diagram for $d<4$.

Note that all these fixed points have $2-dv<0$ for $d>4$, which satisfies the CCFS bound (see Section 3). Note that $\zeta=1/2$ for C at two loops, in agreement with the prediction $\zeta=1/2$ to all orders in ε.

How many strong-disorder fixed points are there? There is certainly one corresponding to initial values of the parameters on the line $u=-\Delta/8$, corresponding to ideal chains localized in a random potential (and if one end is fixed, forming a totally stretched tadpole-like configuration[14]). It is not clear whether there should be only one SAW strong-disorder fixed point as the initial value of U is varied. For instance, at the level of the perturbative flow of Figs. 4 and 5, there seem to be two regimes for $u<u_c(d)$ and $u>u_c(d)$ with $u_c(d)=0$ for $d<4$ and $u_c(d)=|\varepsilon|/32$ for $d=4+|\varepsilon|$ (this might be irrelevant at strong disorder). It would be interesting to develop real-space approaches where the amount of self-avoidance U could be varied (in the present paper $U=\infty$).

Finally, it is interesting to notice from the recursion relations (5.1) that there is an effect of *screening* of the disorder by the self-avoidance, although for long enough chains $N>\xi$ the disorder ultimately wins in $d\leqslant 4$. In $d=4$, this length scale can be particularly large if the ratio u_0/Δ_0 is large. Then $\xi\sim\exp(u_0/\Delta_0^2)$ for $u_0/\Delta_0\gg 1$. This effect may lead to difficulties in observing the asymptotic behavior in numerical simulations.

6. VARIOUS SCENARIOS FOR THE EXPONENTS

In this section we discuss several scenarios leading to predictions for the exponents ζ and ω and relations between them. These predictions are based on qualitative arguments and, indeed, they are to some extent mutually exclusive. Nonetheless, we think they can serve as useful guides. At this stage the most cautious approach would be to conclude that at the strong-disorder fixed point ζ and ω can assume *a priori* arbitrary values consistent with the bounds of Section 2.

Since we will attempt to make some analogies with the better understood problem of the directed walk, it is useful to write both partition functions in the Edwards representation:

$$Z_{\mathrm{DSAW}}(R,N)=\int_{r(0)=0}^{r(N)=R} Dr(t)\exp\left\{-\int dt\left[\frac{1}{4D}\left(\frac{dr}{dt}\right)^2+\Delta^{1/2}V(r(t),t)\right]\right\}$$

$$Z_{\mathrm{SAW}}(R,N)=\int_{r(0)=0}^{r(N)=R} Dr(s)\exp\left\{-\int ds\left[\frac{1}{4D}\left(\frac{dr}{ds}\right)^2\right.\right. \tag{6.1}$$

$$\left.\left.+\frac{U}{2}\int ds'\,\delta(r(s)-r(s'))+\Delta^{1/2}V(r(s))\right]\right\} \tag{6.2}$$

6.1. Flory Arguments Relating ζ, ω, and ζ_0

In the DSAW problem, due to Galilean invariance, D is unrenormalized. More precisely, this is because

$$\log Z(R, N)_V = \log Z(0, N)_{V'} + R^2/4DN$$

where V' is the potential configuration obtained from V via a Galilean transformation $V'(r, t) = V(r - tR/N, t)$ which corresponds to a rotation of small angle R/N. Since V' has the same distribution as V, (14, 36, 37) the kinetic energy (elastic term) scales as *exactly* R^2N^{-1}. Given that the DSAW is controlled by a zero-temperature fixed point with fluctuations in the position of the head of the walk $R(N)$ governed strictly by the fluctuations[5] of the free energy, one has $R^2N^{-1} \sim N^{\chi}$, which provides the exact relation $\chi = 2\zeta_\perp - 1$. It is important to note that being at a zero-temperature fixed point *does not* mean that the elastic entropy can be neglected. The above invariance property shows that there is a cost $R^2/4DN$ for "stretching" the DSAW in the transverse dimension.

Is there an analogous relation for the SAW? It is well known[38] that for a SAW in a pure environment one has

$$\log[Z(R, N)/Z(N)] \sim -(R/N^{\zeta_0})^{1/(1-\zeta_0)} \qquad (RN^{-\zeta_0} \gg 1) \tag{6.3}$$

$$\log[Z(R, N)/Z(N)] \sim -(N/R^{1/\zeta_0}) \qquad (RN^{-\zeta_0} \ll 1) \tag{6.4}$$

which is true at least in the scaling region. Suppose that the effect of the disorder is to stretch the SAW due to the competition between the free energy fluctuations and the stretching energy (e.g., elastic entropy). If one assumes the elastic entropy is the same as for the pure SAW, Eq. (6.3), one finds the relation[6]

$$\omega = \frac{1}{\zeta}\frac{\zeta - \zeta_0}{1 - \zeta_0} \tag{6.5}$$

This relation yields $\chi \equiv \zeta\omega = 2\zeta - 1$ for $d > 4(\zeta_0 = 1/2)$ and also predicts $\zeta = \zeta_0$ if $\omega = 0$. However, we did not find a symmetry argument analogous to Galilean invariance showing that (6.5) is exact. This relation would be exact if, as for the DSAW, one could write the free energy as a sum of a random part whose distribution would be independent of R and a nonran-

[5] Free energy fluctuations are believed to scale as N^{χ} both with respect to the environment and to the position of the head.[14]

[6] There is an interesting equivalent formulation of the relation (6.5) as $\nu' = \nu'_0 \equiv \zeta_0$, where ν' is the exponent characterizing the singularity of the free energy of the SAW. Details are given in Section 7.

dom part equal to the stretching energy of the pure SAW, e.g., $\log Z_N(R) = \log Z_N(R_0) + E_{el}(R/R_0)$.

It is also interesting to note that if disorder was to compress the SAW, the relation suggested by (6.4), $\omega = 1/\zeta - 1/\zeta_0$, is *excluded* by the CCFS bound for $d \leqslant 4$.

From the naive dimension of the disorder term in (6.2) one can extract a second relation between ζ and ω. Since the disorder term scales as $N/R^{d/2}$ we obtain

$$\omega = \frac{1}{\zeta} - \frac{d}{2} \tag{6.6}$$

Combining (6.5) and (6.6) leads to a Flory theory for ζ and ω with the result

$$\zeta_F(d) = \frac{1}{1 + \frac{1}{2}d(1-\zeta_0)} \tag{6.7}$$

and

$$\omega_F(d) = \frac{1}{2}(2 - d\zeta_0) \tag{6.8}$$

giving $\zeta_F(1) = \zeta_0 = 1$, $\zeta_F(2) = 0.8 (\zeta_0 = 0.75)$, $\zeta_F(3) = 0.618$ $(\zeta_0 = 0.588...)$, and $\zeta_F(4) = \zeta_0 = 1/2$ and $\omega_F(1) = 1/2$, $\omega_F(2) = 1/4$, $\omega_F(3) = 0.118$, and $\omega_F(4) = 0$. This Flory estimate is to be compared with the Flory estimate $\zeta_{\perp F} = 3/(4 + d')$ for the DSAW[39] coming from the identification $R^2/N \sim N^{1/2} R^{-d'/2}$ (d' is the number of transverse dimensions). For the DSAW the Flory estimate is not accurate, but still gives a rough idea of how the wandering exponent is changed ($\zeta_{\perp F} = 0.6$ compared to exact value $\zeta_\perp = 2/3$ in $d' = 1$). Here the difference ζ_F/ζ_0 is near unity (within 6%) in qualitative agreement with the RSRG results and one can speculate that (6.7) gives a reasonable approximation on Euclidean lattices. On the other hand, $\chi_F = \omega_F \zeta_F$ for $d = 2$ and 3 is significantly smaller than the corresponding results on hierarchical lattices.

The Flory result for ζ can also be obtained by finding the size R which minimizes a free energy of the form

$$\left(\frac{R}{N^{\zeta_0}}\right)^{1/(1-\zeta_0)} - NR^{-d/2} \tag{6.9}$$

where the first term incorporates the combined effects of self-avoidance and elasticity and the second term represents the disorder. The present Flory

theory is distinguished from the usual Flory arguments[16] because we start at the SAW fixed point rather than the Gaussian fixed point. A similar Flory theory based on the SAW fixed-point theory was proposed recently to predict the size exponent self-avoiding loops under pressure in $d=2$ and seems to be in good agreement with numerical data (this is a problem without disorder).[40]

One objection one can make to this argument is that adding more self-avoidance to (6.9) (i.e., a term of the form N^2R^{-d}) would dominate the disorder term and lead to a modification of ζ. There is no inconsistency if one remembers that the idea here is to make a Flory argument starting from the nontrivial SAW fixed point. First, from the RG recursion relations (5.1) it is clear that at the SAW fixed point, additional self-avoidance *is* irrelevant ($U \to U^*$) and does not change the exponent, whereas disorder is relevant and drives the system to a new strong-disorder fixed point. We are assuming in the Flory theory that disorder is relevant and changes the scaling behavior for the size of the walk.

The above Flory argument can be easily generalized to self-avoiding manifolds (SAM). By equating $(R/N^{\zeta_0})^{1/(1-\zeta_0)} \sim N^D R^{-d/2}$ one finds

$$\zeta_F = \frac{\zeta_0 + D(1-\zeta_0)}{1 + \frac{1}{2}d(1-\zeta_0)}$$

which remains very close to the Flory value $\zeta_{0F} = (D+2)/(d+2)$ for the pure SAM.[41]

The dimensional relation (6.6) is a statement that ω and ζ saturate the CCFS bound (3.2). If we replace (6.6) by the exact bound (3.2) and suppose that there is an additional monotonic relation between ζ and ω, for example (6.5), such that ω is an increasing function of ζ, then we obtain the result that the Flory value is a lower bound for ζ:

$$\zeta \geqslant \zeta_F$$

As was also noticed by D. S. Fisher,[42] there is a bound, analogous to the CCFS bound, for the DSAW. Since for an anisotropic system $\alpha \equiv 2 - \zeta_{||} - d'\zeta_\perp = 1 - d'\zeta_\perp$, the generalization of (3.2) for anisotropic systems is

$$\chi \geqslant \frac{1}{2} - \frac{d'}{2}\zeta_\perp$$

which leads to

$$\zeta_\perp \geqslant \zeta_{\perp F} = \frac{3}{4+d'}$$

a presumably rigorous bound, which is obeyed by known results for DSAWs (it is an equality in $d = d' + 1 = 1$, since $\chi = 1/2$). Note that the isotropic bound (3.2) with $\zeta = \zeta_{||} = 1$ yields only $\chi \geqslant 1 - d'/2$ or $\zeta \geqslant 1 - \frac{1}{4}d'$. An analogous bound should hold for the interface problem.[42]

6.2. Replica Bound States and Scaling Argument

6.2.1. Directed Walk. If the DSAW problem is formulated with replicas, disorder produces an attractive interaction between replicas and they form bound states. For a continuum model in $d' = 1$, it is possible to exactly compute the energy $E_0(k)$ of the bound state of k replicas,[43] from which one obtains for integer k, the leading behavior for large N:

$$\langle Z^k \rangle \sim \langle Z \rangle^k \exp[-NE_0(k)]$$

where $E_0(k) = -c(k^3 - k)$. If one naively continues this expression to any real k, one finds that

$$\langle \exp k(\log Z - \langle \log Z \rangle) \rangle \sim \exp(cNk^3)$$

If one then assumes that the distribution of the variable $\Delta F = \log Z - \langle \log Z \rangle$ takes, in this continuum model, a scaling form

$$\Pr\{\Delta F\} \sim N^{-\chi} f[\Delta F N^{-\chi}] \tag{6.10}$$

one must have $\chi = 1/3$.

Since $E_0(k)$ is known exatly only in $d' = 1$, one can try to estimate it for any d'. A simple argument, first given by Zhang[44] in the context of the codimension-1 interface problem (which coincides with the DSAW for $d' = 1$), allows us to estimate $E_0(k) \sim k^\beta$ for large k. Then, if the scaling form (6.10) holds, one must have $\langle \exp(k\Delta F) \rangle \sim \exp(cNk^{1/\chi})$ and $\chi = 1/\beta$. This identification is reasonable only if the behaviors for large and small k are the same, which holds if the large fluctuations of the free energy scale in the same way as the typical fluctuations [e.g., if there is only one length scale in (6.10), which might be true since it is a continuum model]. Although this assumption works for $d' = 1$, there is no general justification for it.

The energy of a bound state of k replicas of transverse extension r can be estimated as

$$E(k) \sim kD\frac{N}{r^2} - \Delta k^2 \frac{N}{r^{d'}} \tag{6.11}$$

which is minimized for $r \sim (\Delta D^{-1} k)^{-1/(2-d')}$ and thus $E_0(k) \sim D(\Delta D^{-1})^{2/(2-d')} k^{(4-d')/(2-d')}$, leading to

$$\chi = \frac{2-d'}{4-d'}, \qquad \zeta = \frac{3-d'}{4-d'}$$

These values are *exact* in $d = d' + 1 = 1$ and $d = 2$. They predict $\chi = 0$ and $\zeta = 1/2$ in $d = 3$ (for $d' > 2$, since weak disorder is not relevant, there is not a continuum model describing the weak-disorder limit of a discrete model, e.g., one has to consider a discrete model).

There is another way to formulate the above argument without introducing replicas. The basic remark is that the same result can be obtained from a dimensional analysis of the Fokker–Planck equation:

$$\frac{\partial}{\partial N} Z = D \nabla^2 Z - VZ$$

Imposing the condition that the two last terms scale in the same way leads to a time-dependent length scale such that $DL^{-2} \sim (\Delta)^{1/2} L^{-d'/2} T^{-1/2}$ or equivalently $L \sim (D^2 T/\Delta)^{1/(4-d')}$. Then one estimates the free energy fluctuations by equating the right-hand side to the left-hand side of the above formula with $Z \sim \exp(\Delta F)$, which gives $\Delta F \sim D(D^2/\Delta)^{-2/(4-d')} \times N^{(2-d')/(4-d')}$ as above.

6.2.2. Self-Avoiding Walk. A similar argument can be made to estimate the bound-state energy of replicas for the self-avoiding walk. As was discussed in ref. 5, the optimal configuration, in the regime described by a continuum field theory (e.g., for $k < k_c$ at which discretization effects become relevant—here k_c can be taken infinite since small disorder is relevant for $d < 4$), is a self-avoiding sausage in which all the k replicated mutually attracting SAWs are confined. The width r of the sausage can be estimated as in (6.11) from the energy:

$$E \sim k \frac{N}{r^{1/\zeta_0}} + \Delta k^2 \frac{N}{r^{d-1/\zeta_0}} \tag{6.12}$$

where the first term is the sum of the confinement energy of each SAW inside the sausage, and the second term is the interaction energy of the SAWs, precisely $\sim N^2/(r^{d-1} r_{\|})$, where $r_{\|} \sim N r^{1-1/\zeta_0}$ is the length of the sausage, as given by a blob argument. Minimization of (6.12) gives $r \sim (\Delta k)^{-\zeta_0/\alpha_0}$ (for $d < 4$), and $E \sim \Delta^{1/\alpha_0} N k^{(1+\alpha_0)/\alpha_0}$. This gives a new prediction:

$$\chi = \frac{\alpha_0}{1+\alpha_0} \tag{6.13}$$

a very general result, valid for SAWs or DSAWs, if α_0 denotes the value of the pure-system specific heat exponent $\alpha_0 = 2 - d\zeta_0$ ($\alpha_0 = 1 - d'/2$ for the DSAW, see above). Equation (6.13) is exact for the DSAW with $d' = 0$ and seems to provide a good approximation for the SAW in $d = 2$, where one finds $\chi(d=2) = 1/3$, quite close to the results on "$d = 2$" hierarchical lattices *A*, *B*, and *E*. The result $\chi(d=3) = 0.190$ is much smaller than the "$d = 3$" hierarchical lattice *F*.

The fact that, in this argument, the center-of-mass motion for the SAW is itself a self-avoiding walk does not indicate *a priori* that $\zeta = \zeta_0$. The same phenomenon occurs for the DSAW for which the center-of-mass motion is itself an unperturbed random walk, as a consequence of Galilean invariance. In that case the consequence was that an elasticity theory applies, leading to a relation between χ and ζ. A similar analysis for the SAW has been attempted in Section 6.1.

It was observed in ref. 5 that the length scale b^* (see Section 5.1) appearing in the field theory analysis of the first-order transition of replicas $k \geqslant 2$ was *identical* to first order in ε to the confinement length r resulting from the naive minimization argument (6.12). Here, we make the observation that the result of the field theory analysis for the replica moments for $k \geqslant 1 + O(\varepsilon)$, namely[5]

$$\langle Z^k \rangle \sim \exp(c_1 kN + c_2 \varepsilon^{1-4/\varepsilon} \Delta^{4/\varepsilon} k^{1+4/\varepsilon} N)$$

actually suggests that

$$\chi = \frac{\varepsilon/4}{1+\varepsilon/4}$$

to lowest order in ε, identical to (6.13) since $\alpha_0 = \varepsilon/4 + O(\varepsilon^2)$. This result is not fully justified, since it depends on the continuation to $k \to 0$, while perturbation theory breaks down close to $k = 1$. It appears that the first-order transition in the replica moments signals the existence of bound states in the theory, the energy of the bound state determined by the transition temperature of the first-order transition. This should be a general feature of disordered systems, and an investigation of its consequences for the DSAW perturbatively around $d = 2$ would be interesting.

Finally, note that if $\zeta \geqslant \zeta_0$ the prediction (6.13), $\chi = \alpha_0/(1+\alpha_0)$, obeys the CCFS bound (3.2) and saturates it in $d = 1$.

6.3. Alternative Flory Theories

An second Flory theory is based on the ideal chain and begins with a free energy of the form

$$R^2/N + UN^2 R^{-d} - \Delta^{1/2} N^{1/2} R^{-d/2}$$

The first and second terms represent the elastic and self-avoidance energies, respectively. The third term is the typical energy of the random potential (in naive dimensions). Since the third term scales as $N^{\alpha_0/2} \ll N^{\alpha_0}$ ($\alpha_0 \equiv 2 - d\zeta_0$), it is always much smaller than the self-avoidance energy, and one concludes that disorder does not affect the size of the chain

$$\zeta = \zeta_0 \tag{6.14}$$

On the other hand, the fluctuations in the free energy are controlled by the last term and one obtains

$$\chi = 1 - d\zeta_0 = \alpha_0/2 \tag{6.15}$$

Equations (6.14) and (6.15) are consistent with the Lifshitz calculation[5] of $\langle Z^k \rangle$ for $k < 1$ and are in reasonable agreement with the RSRG approaches for $d = 2$ and 3. On the other hand, the assumption that self-avoidance dominates over disorder is hard to reconcile with the ideas of a zero-temperature fixed point where disorder is the dominant effect.

Finally, we review Flory ideas presented by Obukhov. In ref. 7 he proposes the elasticity relation $\chi = 2\zeta - 1$. This relation replaces our equation (6.5) and is isomorphic to the exact relation for the DSAW. It is obtained using the ideal chain result for the stretching energy, R^2/N. One justification would be to remember the Flory argument for the pure SAW (e.g., minimization of the sum $R^2/N + N^2/R^d$) and then to argue that if a SA chain is highly stretched (e.g., by disorder), then self-avoidance is irrelevant and the only elastic energy comes from the term R^2/N. This form of the stretching energy cannot hold near $d = 1$, or near $\zeta = 1$, since it predicts $\chi = 1$ instead of the correct value $\chi = 1/2$. By comparison, (6.5) is plausible even when self-avoidance is relevant. Obukhov argues that $\chi = 2\zeta - 1$ holds only if $\zeta > \zeta_0$. It seems to us, however, that there might be an inconsistency with the approach of ref. 7, since self-avoidance *cannot* be irrelevant to stretching the walk. It is known that in the absence of self-avoidance the ideal chain has a quite different shape (i.e., a tadpole configuration with a collapsed head and a stretched tail[35]).

7. CROSSOVER FROM SAW TO DSAW IN A RANDOM POTENTIAL

In this section we analyze the crossover from the SAW to the DSAW in a quenched random potential, using simple scaling arguments. This crossover also exists for pure SAWs, but is more subtle here because disorder rather than thermal fluctuations dominates the physics. There are two ways to observe this crossover: (1) One can impose a constant stretching

force F on the free end of a SAW of N steps (the other end being fixed at the origin). (2) One can fix both ends at distance L from each other, introduce a chemical potential per step, and study the stretched limit $(\mu - \mu_c) L^{1/\nu} \gg 1$. We first study the latter problem using hierarchical lattices as a guide. Our results are probably also true on Euclidean lattices. In this section we denote by $\chi' = \omega'$ the free energy fluctuation exponent of the DSAW on the same lattice.

7.1. Singularity of the SAW Free Energy and Its Fluctuations at $\mu = \mu_c$

It is clear by looking at the recursion relations (2.1) and (2.5) on hierarchical lattices that if one starts from a small $\mu - \mu_c > 0$, one flows away from the SAW strong-disorder fixed point toward the DSAW fixed point, since for large and positive f_i the recursion relation reduce to the corresponding relations on the directed sublattice obtained by keeping only the shortest paths (this is true both for the finite and zero-temperature recursion relations). The large limit L with $\mu - \mu_c$ fixed thus corresponds to DSAWs.

The first consequence is that the free energy is extensive, as was demonstrated in refs. 23 and 24, and that

$$f(\mu) = \lim_{L \to \infty} L^{-1} \log Z_L(\mu, e)$$

exists and is independent of the environment e for $\mu > \mu_c$. We thus begin by showing that, as a consequence of a scaling assumption, the free energy of a SAW in a quenched random environment has a singularity $f(\mu)_{\text{sing}} \sim (\mu - \mu_c)^{\nu'}$ at the critical fugacity $\mu = \mu_c$ with

$$\nu' = \frac{1 - \omega}{1/\zeta - \omega} \tag{7.1}$$

an expression which reduces to $\nu' = \nu \equiv \zeta$ for the pure case, $\omega = 0$, of a SAW or DSAW ($\zeta \equiv \zeta_{||} = 1$).

Let us first recall the usual scaling argument for the *pure* SAW. As is well known,

$$Z(L, N) \sim N^{\gamma - 1 - \zeta_0 d} \exp(\mu_c^0 N)\, F(LN^{-\zeta_0})$$

Upon performing the Laplace transform over N and neglecting the power-law corrections, one deduces the scaling form $\log Z_L(\mu) \sim f[(\mu - \mu_c) L^{1/\zeta_0}]$. Since the free energy has to be extensive $\sim L$ (an obvious

fact on hierarchical lattices, first demonstrated on Euclidean lattice in ref. 38), one must have

$$f_{\text{sing}} = \lim_{L \to \infty} L^{-1} \log Z_L(\mu) \sim (\mu - \mu_c)^{\zeta_0}$$

This is the equivalent of the hyperscaling relation in an ordinary thermodynamic system. Note that the inclusion of power-law corrections does not modify this result (see ref. 38).

In the strong-disorder regime of the SAW in a random environment one expects $\log Z(L, N)$ to have a wide distribution, which we symbolically write

$$\log Z(L, N) \sim L^{\omega} F[NL^{-1/\zeta}, e] \tag{7.2}$$

which means that the distribution of $L^{-\omega} \log Z$ with respect to the configuration e depends on N and L only through the scaling variable $NL^{-1/\zeta}$. Although it is more difficult here to make a precise argument, it is clear that upon Laplace transform over N, the saddle point will satisfy

$$(\mu - \mu_c) N \sim L^{\omega}$$

in the scaling region, which shows that at a strong-disorder fixed point $(\mu - \mu_c)$ *does not* scale like N^{-1}. Instead, from (7.2) we find the scaling form

$$\log Z(L, \mu, e) \sim L^{\omega} G[(\mu - \mu_c) L^{1/\nu}, e] \tag{7.3}$$

with $1/\nu = 1/\zeta - \omega$, in agreement with the analysis of Section 3 leading to the identification of the exponent ν, and with the exact analysis in $d = 1$. Indeed one can expand (7.3) around $\mu = \mu_c$ and find

$$\log Z(L, \mu, e) = \log Z(L, \mu_c, e) + a(e)(\mu - \mu_c) L^{1/\zeta} + \cdots$$

with $\log Z(L, \mu_c, e) \sim c(e) L^{\omega}$ and $\bar{N}(L, e) \sim a(e) L^{1/\zeta}$. Now the requirement of extensivity for large L implies the result (7.1) for the free energy singularity exponent. Note that the distribution of $L^{-1} \log Z(L, \mu, e)$ goes to a delta function independent of e.

One can go further and use a matching condition from the SAW onto the DSAW fixed point. Let us call $r = L(\mu - \mu_c)^{\nu}$ the scaling variable entering (7.3) and suppose that for large r, the distribution of $g = L^{-\omega} \log Z(\mu, L, e)$ goes to a fixed distribution of width $\sim r^{v}$ centered around $g = g_0 \sim cr^{u}$, where u and v are two exponents to be determined by matching. Since we know that at the fixed point for the DSAW

$\log Z \sim fL + b(e)\, L^{\omega'}$, one must have $u + \omega = 1$ and $v + \omega = \omega'$, which implies that the free energy of the SAW has the asymptotic scaling form:

$$\log Z(L, \mu, e) \sim L(\mu - \mu_c)^{\nu'} + d(e)\, L^{\omega'}(\mu - \mu_c)^{\nu(\omega' - \omega)} + \cdots$$

where $d(e)$ is of order unity and has a fixed distribution.

Note that the above analysis is valid everywhere in the domain of strong disorder, e.g., outside the immediate vicinity of the pure fixed point (or the weak-disorder phase). More precisely, when small disorder is relevant, it is valid for $(\mu - \mu_c) \ll \Delta^{\lambda/\nu}$, where λ is the eigenvalue of the disorder at the pure fixed point and Δ its strength.

Note that on a hierarchical lattice on the other side of the transition $(\mu - \mu_c) < 0$ one has the directed problem of the *longest* possible paths; thus ν' is unchanged, but the exponent of the fluctuations $\nu(\omega' - \omega)$ is changed since ω' is changed.

7.2. SAW Submitted to a Stretching Force and Blob Arguments

As is well known, one can give simple "blob" arguments to describe the behavior of a self-avoiding walk in the presence of a small stretching force F acting on its free end. The SAW is unperturbed on scales smaller than L_0 defined by $FL_0 \sim T = 1$. The SAW can be seen as composed of blobs themselves forming a pure DSAW, and thus $L = (N/N_0)\, L_0 \sim F^{-1 + 1/\zeta_0} N$. Note that the same result is obtained by balancing the stretching energy $(L/N^{\zeta_0})^{1/(1-\zeta_0)}$ with FL.

In the presence of disorder, the SAW in the strong-disorder regime is unperturbed by the stretching force $F \ll 1$ on scales smaller than L_1, where $FL_1 \sim N_1^{\chi} \sim L_1^{\omega}$. This is because temperature is irrelevant and only free energy fluctuations are important. This gives $L_1 = F^{-1/(1-\omega)}$ ($\gg L_0$, which is replaced by L_1). On length scales larger than L_1 the SAW has the shape of a directed string of blobs of size L_1 along the direction of F. Then, the properties of the DSAW can be used. Thus, $L \sim (N/N_1)\, L_1$ with $N_1 \sim L_1^{1/\zeta} \sim F^{-1/(\zeta - \chi)}$, which gives for the extension of the DSAW along the direction of the force

$$L \sim N F^{(1-\zeta)/(\zeta - \chi)}$$

The free energy f of the stretched SAW and its typical fluctuation Δf can also be estimated by the same blob arguments. The free energy is $f \sim FL$ and the fluctuation Δf is given by the free energy fluctuations of a DSAW of N/N_1 steps; with a renormalized disorder strength L_1^{ω} at the scale L_1 we find

$$f \sim N F^{(1-\chi)/(\zeta - \chi)}, \qquad \Delta f = N^{\chi'} F^{(\chi' - \chi)/(\zeta - \chi)}$$

7.3. New Argument for Exponent Relation for the DSAW

Clearly the above argument must itself apply to a directed walk. Suppose the DSAW is directed in the t direction (eventually as a result of applying a large force to an isotropic SAW) and one applies an additional small force F in the x direction to the head of the polymer. Then, from the blob argument one has $x \sim tF^{(1-\zeta_\perp)/(\zeta_\perp - \chi)}$. But applying those two forces is like applying to an isotropic system a total force in the direction making a small angle $\sim F$ with the t axis. From linear response (e.g., the fact that the displacement must be along the direction of the total applied force) this implies $x/t \sim F$ and thus we find

$$\chi = 2\zeta_\perp - 1$$

This seems to be a new argument for justifying the exponent relation, which seems to rely on linear response and rotational invariance. Note that one finds $f = NF^2$, which is also natural, since f should be even in F.

Finally, note that the elasticity relation (6.5) amounts to the hypothesis that the response exponent φ such that $L \sim F^\varphi$ is the same for a SAW with disorder as for the pure SAW.

8. CONCLUSION

We have studied the problem of self-avoiding walks in a quenched random environment. We have investigated SAWs on hierarchical lattices using an exact renormalization group method and shown that this problem is controlled by a strong-disorder fixed point in low dimension or for sufficiently large disorder. The existence of a strong-disorder fixed point on hierarchical lattices strongly suggests that similar behavior holds on Euclidean lattices, in agreement with conclusions[7] based upon runaway trajectories in the field theory describing SAWs in random environments. The strong-disorder fixed point is characterized by two new exponents, ζ and ω. We find that the size exponent ζ for "two-dimensional" hierarchical lattices is slightly larger than the corresponding pure exponent on the same lattice. It remains an open question whether ζ equals 3/4 for a disordered two-dimensional Euclidean lattice as suggested by recent simulations[10] or whether ζ is slightly larger than 3/4.

In addition to accurate values of the exponents on hierarchical lattices, we have obtained more general relations and bounds based upon the supposition of a strong-disorder fixed point. We find that $1/\zeta - \omega \leqslant d/2$ and that the quenched critical fugacity is strictly greater than the annealed critical fugacity.

We have obtained a variety of estimates for ζ and ω based upon

dimensional analysis, scaling arguments, and analogies to directed walks. These estimates are of the form that ζ and ω are simple functions of the dimension and the pure exponent ζ_0. All of these approaches predict $\zeta \geqslant \zeta_0$ and $1/2 > \omega > 0$ in the range $1 < d < 4$; however, it is difficult to judge which is closer to capturing the key features of the problem. It may be that the validity of the approximate methods can be tested by numerical simulations of the exponent ω, since the different approaches yield rather different values of this exponent.

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